

# BLOWING UP FINITELY SUPPORTED COMPLETE IDEALS IN A REGULAR LOCAL RING

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**ABSTRACT.** Let  $I$  be a finitely supported complete  $\mathfrak{m}$ -primary ideal of a regular local ring  $(R, \mathfrak{m})$ . We consider singularities of the projective models  $\text{Proj } R[It]$  and  $\text{Proj } \overline{R[It]}$  over  $\text{Spec } R$ , where  $\overline{R[It]}$  denotes the integral closure of the Rees algebra  $R[It]$ . A theorem of Lipman implies that the ideal  $I$  has a unique factorization as a  $*$ -product of special  $*$ -simple complete ideals with possibly negative exponents for some of the factors. If  $\text{Proj } \overline{R[It]}$  is regular, we prove that  $\text{Proj } R[It]$  is the regular model obtained by blowing up the finite set of base points of  $I$ . Extending work of Lipman and Huneke-Sally in dimension 2, we prove that every local ring  $S$  on  $\text{Proj } \overline{R[It]}$  that is a unique factorization domain is regular. Moreover, if  $\dim S \geq 2$  and  $S$  dominates  $R$ , then  $S$  is an infinitely near point to  $R$ , that is,  $S$  is obtained from  $R$  by a finite sequence of local quadratic transforms.

## 1. INTRODUCTION

Let  $(R, \mathfrak{m})$  be a regular local ring of dimension at least 2. A regular local ring  $S$  that dominates  $R$  is *infinitely near to  $R$*  if  $\dim S \geq 2$  and  $S$  may be obtained from  $R$  by a finite sequence (possibly empty) of local quadratic transforms. An infinitely near point  $S$  to  $R$  is a *base point* of an ideal  $I$  of  $R$  if the transform  $I^S$  of  $I$  in  $S$  is a proper ideal of  $S$ . The set of base points of an ideal  $I$  of  $R$  is denoted  $\mathcal{BP}(I)$ , and the ideal  $I$  is said to be *finitely supported* if the set  $\mathcal{BP}(I)$  is finite.

The infinitely near points to  $R$  form a partially ordered set with respect to domination. The regular local ring  $R$  is the unique minimal point with respect to this partial order. For an ideal  $I$  of  $R$ , the set  $\mathcal{BP}(I)$  of base points of  $I$  is a partially ordered subset of the set of infinitely near points to  $R$ . If the set  $\mathcal{BP}(I)$  is finite, we refer to the maximal regular local rings in  $\mathcal{BP}(I)$  as *terminal base points* of  $I$ . If  $I$  is a finitely supported ideal, then results of Lipman [10, Prop. 1.21, Cor. 1.22] imply that  $\dim S = \dim R$  and  $S/I^S$  is Artinian for each base point  $S$  of  $I$ . In particular, the ideal  $I$  is  $\mathfrak{m}$ -primary.

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**Definition 1.1.** Let  $(R, \mathfrak{m})$  be a regular local ring and let  $I$  be a finitely supported  $\mathfrak{m}$ -primary ideal. Let  $\Gamma := \mathcal{BP}(I)$  denote the finite set of base points of  $I$ . By successively blowing up the maximal ideals of the points in  $\Gamma$  we obtain a regular projective model<sup>1</sup>  $X_\Gamma$  over  $R$  and a projective morphism  $X_\Gamma \rightarrow \operatorname{Spec} R$ . We call  $X_\Gamma$  the *saturated regular model* associated to the ideal  $I$ , or more precisely, to the set  $\Gamma = \mathcal{BP}(I)$ .

The model  $X_\Gamma$  may be obtained by first blowing up the maximal ideal  $\mathfrak{m}$  of  $R$  to obtain the regular model  $\operatorname{Proj} R[\mathfrak{m}t] = X_1$ . Each infinitely near point  $S$  in  $\Gamma = \mathcal{BP}(I)$ , other than  $R$ , dominates a unique point on the model  $X_1$ . The points in  $\Gamma$  in the first neighborhood of  $R$  are obtained from  $R$  by one local quadratic transform and are points on the model  $X_1$ . Each infinitely near point  $S$  in  $\Gamma \setminus \{R\}$  is either a point on the model  $X_1$  or is an infinitely near point to a unique point  $S_1$ , where  $S_1$  is a point on the model  $X_1$ . Associated to each infinitely near point  $S_1 \in X_1$  such that  $\dim S_1 = \dim R$ , there exists a unique coherent  $\mathcal{O}_{X_1}$ -ideal sheaf  $\mathcal{I}$  such that the stalk  $\mathcal{I}_{S_1}$  is the maximal ideal of  $\mathcal{O}_{X_1, S_1}$  and the stalk  $\mathcal{I}_T = \mathcal{O}_{X_1, T}$  for each point  $T$  in  $X_1 \setminus \{S_1\}$  [10, Lemma 2.3].

On  $X_1$ , we blow up the ideal sheaf that is the product of the ideal sheaves that correspond to the points  $S_1 \in \Gamma \cap X_1$  to obtain the regular model  $X_2$ . There exist associated projective morphisms  $X_2 \rightarrow X_1 \rightarrow \operatorname{Spec} R$ . We continue this process to obtain the regular model  $X_\Gamma$  and projective morphism  $X_\Gamma \rightarrow \operatorname{Spec} R$  in which each of the infinitely near points in  $\Gamma$  has been blown up.

Let  $S$  be a regular local ring infinitely near to  $R$ . If  $S \neq R$ , then  $S$  is infinitely near to a point on  $\operatorname{Proj} R[\mathfrak{m}t]$ . More generally, if  $S$  is a regular local ring infinitely near to  $R$ , then  $S$  is infinitely near to a point on  $X_\Gamma$  if and only if  $S$  is not a base point of  $I$ , that is, if and only if  $S \notin \Gamma$ .

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<sup>1</sup> We are using the language of Section 17, Chapter VI of Zariski-Samuel [17]. Thus, for  $R$  a subring of a field  $K$  and  $A$  a finitely generated  $R$ -subalgebra of  $K$ , the *affine model* over  $R$  associated to  $A$  is the set of local rings  $A_P$ , where  $P$  varies over the set of prime ideals of  $A$ . A *model*  $M$  over  $R$  is a subset of the local subrings of  $K$  that contain  $R$  that has the properties: (i)  $M$  is a finite union of affine models over  $R$ , and (ii) each valuation ring of  $K$  that contains  $R$  dominates at most one of the local rings in  $M$ . This second condition is called *irredundance*. A model  $M$  over  $R$  is said to be *complete* if each valuation ring of  $K$  that contains  $R$  dominates a local ring in  $M$ . The model  $M$  is said to be *projective* over  $R$  if there exists a finite set  $a_0, a_1, \dots, a_n$  of nonzero elements of  $K$  such that  $M$  is the union of the affine models defined by the rings  $A_i = R[\frac{a_0}{a_i}, \frac{a_1}{a_i}, \dots, \frac{a_n}{a_i}]$ ,  $i = 0, 1, \dots, n$ . The models we consider are either affine or projective models over a Noetherian integral domain  $R$ . In the language of schemes, these models correspond, respectively, to affine or projective schemes over  $\operatorname{Spec} R$ .

The model  $X_\Gamma$  and projective morphism  $X_\Gamma \rightarrow \operatorname{Spec} R$  have the property that for each finitely supported ideal  $J$  of  $R$  such that  $\mathcal{BP}(J) \subseteq \Gamma$ , there exists a morphism  $f : X_\Gamma \rightarrow \operatorname{Proj} \overline{R[Jt]}$  that gives the following commutative diagram:

$$(1.1) \quad \begin{array}{ccc} X_\Gamma & & \\ \downarrow f & \searrow & \\ \operatorname{Proj} \overline{R[Jt]} & \longrightarrow & \operatorname{Spec} R. \end{array}$$

**Remark 1.2.** Let  $R$  be a regular local ring with  $\dim R \geq 2$ . A finite set  $\Gamma$  of points infinitely near to  $R$  is the set of base points of a finitely supported ideal of  $R$  if and only if the set  $\Gamma$  satisfies the following conditions.

- (1)  $R \in \Gamma$ ,
- (2) For each  $S \in \Gamma$ , we have  $\dim S = \dim R$ , and
- (3) For each  $S \in \Gamma$ , each of the regular local rings in the unique chain of local quadratic transforms from  $R$  to  $S$  is in  $\Gamma$ .

Fix a finite set  $\Gamma$  of infinitely near points to  $R$  that satisfies these 3 conditions. For each  $R_i \in \Gamma$ , let  $P_i$  denote the special  $*$ -simple ideal associated to the pair  $R \prec R_i$  [10, Prop. 2.1]. Setting  $I = \prod_{R_i \in \Gamma} P_i$ , we have  $\mathcal{BP}(I) = \Gamma$  and  $\operatorname{Proj} R[It] = X_\Gamma$ .

**Definition-Remark 1.3.** Let  $I$  be a finitely supported ideal of a regular local ring  $R$ . We say the morphism  $f : X_\Gamma \rightarrow \operatorname{Proj} \overline{R[It]}$  of Diagram 1.1 is *biregular at*  $S \in \operatorname{Proj} \overline{R[It]}$  if  $f^{-1}(S) = \{S\}$ .

We say that the ideal  $I$  has a *saturated factorization* if  $\operatorname{Proj} \overline{R[It]}$  is the regular model  $X_\Gamma$ , where  $\Gamma := \mathcal{BP}(I)$ . In the case where  $\dim R = 2$ , this terminology is equivalent to that used in [4, Def. 5.11]. As observed in Remark 1.2, there exist finitely supported ideals  $I^*$  such that  $\mathcal{BP}(I^*) = \mathcal{BP}(I)$  and  $\operatorname{Proj} R[I^*t] = X_\Gamma$ .

In Section 2 we obtain in Corollary 2.8 the following result:

**Theorem 1.4.** *Let  $I$  be a finitely supported  $\mathfrak{m}$ -primary ideal of a regular local ring  $(R, \mathfrak{m})$ . If  $\operatorname{Proj} \overline{R[It]}$  is regular, then  $I$  has a saturated factorization, that is the morphism  $f : X_\Gamma \rightarrow \operatorname{Proj} \overline{R[It]}$  of Diagram 1.1 is an isomorphism. More generally, if each local ring  $S \in \operatorname{Proj} \overline{R[It]}$  is a unique factorization domain, then  $\operatorname{Proj} \overline{R[It]}$  is regular and  $I$  has a saturated factorization.*

**Remark 1.5.** Classical results of Zariski and Abhyankar imply Theorem 1.4 in the case where  $\dim R = 2$ , cf. [4, Prop. 5.12]. The assertion about unique factorization domains is also known in the case of dimension 2 [9, Prop. 3.1] and [8, Cor. 1.2]

We follow the notation of [10]. For a Noetherian local domain  $(R, \mathfrak{m})$ , we denote by  $\text{ord}_R$  the order function defined by the powers of  $\mathfrak{m}$ . If  $R$  is a regular local ring, or more generally if the associated graded ring of  $R$  with respect to  $\mathfrak{m}$  is a domain, then  $\text{ord}_R$  defines a rank 1 discrete valuation on the field of fractions of  $R$ . For an ideal  $I$  in a Noetherian domain, we denote by  $\text{Rees } I$  the Rees valuations of  $I$ . For ideals  $I$  and  $J$  of an integral domain  $R$ , their  $*$ -product, denoted  $I * J$ , is the integral closure  $\overline{IJ}$  of their ordinary product. For an ideal  $I$  of a UFD  $R$  with  $\text{ht } I \geq 2$  and  $S$  a UFD overring of  $R$  with the same field of fractions, the *transform* of  $I$  in  $S$ , denoted  $I^S$ , is  $a^{-1}IS$ , where  $aS$  is the smallest principal ideal of  $S$  containing  $IS$ .

**Discussion 1.6.** Let  $I$  be a finitely supported complete  $\mathfrak{m}$ -primary ideal of a regular local domain  $(R, \mathfrak{m})$  and let  $\Gamma := \mathcal{BP}(I)$ . We have:

- (1) If  $\dim R = 2$ , then associated to each simple complete ideal factor  $J_i$  of  $I$ , there exists an infinitely near point  $R_i \in \Gamma$  such that  $J_i = P_{RR_i}$  is the special  $*$ -simple ideal associated to the pair  $R \prec R_i$  [10, Prop. 2.1]. Furthermore,  $\text{Rees } J_i = \{\text{ord}_{R_i}\}$  (cf. [16, Prop. 14.4.10]), so the Rees valuation rings of  $I$  are in one-to-one correspondence with the distinct simple factors of  $I$  as a product of simple complete ideals (cf. [16, Prop. 10.4.8]).
- (2) In the case where  $\dim R \geq 3$ ,  $\text{Rees } I \subseteq \{\text{ord}_{R_i} \mid R_i \in \Gamma\}$  [5, Prop. 4.3]. A special  $*$ -simple ideal  $P_{RR_i}$  associated to an infinitely near point  $R_i$  contains  $\text{ord}_{R_i}$ , but it often contains additional Rees valuation rings  $\text{ord}_{R_j}$  with  $R_j \in \Gamma$  and  $R_j \neq R_i$ .
- (3) If  $I$  has saturated factorization, then  $\text{Rees } I = \{\text{ord}_{R_i} \mid R_i \in \Gamma\}$ . If  $\dim R = 2$ , then this property characterizes the ideals with a saturated factorization. However, if  $\dim R \geq 3$ , then Item 2 allows the construction of finitely supported complete ideals  $I$  such that  $\text{Rees } I = \{\text{ord}_{R_i} \mid R_i \in \Gamma\}$  and yet  $I$  does not have a saturated factorization, that is, there exist normal local domains  $S \in \text{Proj } \overline{R[It]}$  that are singular, see Example 6.6.
- (4) Assume that  $\text{ord}_{R_i} \in \text{Rees } I$  for each  $R_i \in \Gamma$ . We prove in Theorem 3.2 that the morphism  $f : X_\Gamma \rightarrow \text{Proj } \overline{R[It]}$  of Diagram 1.1 is biregular at each  $S \in \text{Proj } \overline{R[It]}$  that has torsion divisor class group. Therefore for each

singular point  $S \in \text{Proj } \overline{R[It]}$ , the divisor class group of  $S$  is nontorison. Thus the singular local domains in examples such as Example 6.6 must have nontorison divisor class group.

## 2. REGULAR BLOWUP IMPLIES SATURATED FACTORIZATION

In Theorem 2.5, we prove that a local UFD on the blowup of a finitely supported ideal is regular. Lemma 2.1 and Corollary 2.2 are special cases of Lemma 2.4, where we prove that a UFD  $S$  on the blow-up of on a finitely supported  $\mathfrak{m}$ -primary ideal  $I$  has the property that  $\mathfrak{m}S$  is principal. It follows that  $S$  dominates a unique point on  $\text{Proj } R[\mathfrak{m}t]$ , allowing us to apply an inductive argument on the number of base points of  $I$ .

**Lemma 2.1.** *Let  $(R, \mathfrak{m})$  be a regular local domain, let  $I$  be an  $\mathfrak{m}$ -primary ideal, and let  $S$  be a UFD on  $\text{Proj } \overline{R[It]}$  that dominates  $R$ . If there exists a DVR  $V$  that dominates  $S$  such that  $\mathfrak{m}V$  is the maximal ideal of  $V$ , then  $\mathfrak{m}S$  is principal. It follows that  $S$  dominates a local ring on  $\text{Proj } R[\mathfrak{m}t]$ .*

*Proof.* Since  $I$  is  $\mathfrak{m}$ -primary and  $S \in \text{Proj } \overline{R[It]}$ , the ideal  $\mathfrak{m}S$  has height one. Since  $S$  is a UFD,  $\mathfrak{m}S = \rho J$ , where  $\rho$  is a nonzero nonunit of  $S$  and either  $J = S$  or  $J$  is an ideal of  $S$  with  $\text{ht } J \geq 2$ . Since  $V$  dominates  $S$  and  $\mathfrak{m}V$  is the maximal ideal of  $V$ , we must have  $\mathfrak{m}V = \rho V$  and  $J = S$ .  $\square$

**Corollary 2.2.** *Let  $(R, \mathfrak{m})$  be a regular local domain and let  $V = \text{ord}_R$ . Let  $I$  be an  $\mathfrak{m}$ -primary ideal and let  $S$  be the local domain on  $\text{Proj } \overline{R[It]}$  dominated by  $V$ . If  $S$  is a UFD, then  $S = V$  and  $V \in \text{Rees } I$ .*

*Proof.* Lemma 2.1 implies that  $S$  dominates a local ring on  $\text{Proj } R[\mathfrak{m}t]$ . Since  $V = \text{ord}_R \in \text{Proj } R[\mathfrak{m}t]$  and  $V$  dominates  $S$ , we have  $S = V$ .  $\square$

Lemma 2.3 compares the Rees valuations of an ideal  $I$  with those of the transforms of  $I$  in the first neighborhood of  $R$ .

**Lemma 2.3.** *Let  $(R, \mathfrak{m})$  be a regular local domain with  $\dim R \geq 2$  and let  $I$  be an  $\mathfrak{m}$ -primary ideal that has only finitely many base points  $R_1, \dots, R_n$  in the first neighborhood of  $R$ . Then we have*

$$(1) \quad \{\text{ord}_R\} \cup \text{Rees } I = \{\text{ord}_R\} \cup \bigcup_{i=1}^n \text{Rees } I^{R_i},$$

where  $I^{R_i}$  denotes the transform of  $I$  in  $R_i$ . Furthermore, we have

$$(2) \quad \text{Rees } \mathfrak{m} I = \text{Rees } \mathfrak{m} \cup \text{Rees } I = \{\text{ord}_R\} \cup \text{Rees } I.$$

*Proof.* Let  $R'$  be a base point of  $I$  in the first neighborhood of  $R$ . If  $\dim R' < \dim R$ , then there exist infinitely many regular local rings  $S$  in  $\text{Proj } R[\mathfrak{m}t]$  such that  $S \subset R'$ . Notice that  $R' = S_{\mathfrak{m}_{R'} \cap S}$ , where  $\mathfrak{m}_{R'}$  is the maximal ideal of  $R'$ . Since taking transforms is transitive [10, Prop. 1.5.iv] and  $R'$  is a base point of  $I$ , the ring  $S$  is also a base point of  $I$ . Thus the hypothesis that  $I$  has only finitely many base points in the first neighborhood of  $R$  implies that  $\text{ht } I^{R_i} = \dim R_i = \dim R$  for each  $i \in \{1, \dots, n\}$ .

The  $\supseteq$  inclusion of Equation 1 follows from [6, Prop. 3.11].

To see the  $\subseteq$  inclusion of Equation 1, let  $V \in \text{Rees } I$ , and let  $R'$  denote the local quadratic transform of  $\mathfrak{m}$  along  $V$ . If  $IR'$  is not principal, then  $R'$  is a base point of  $I$ , so  $R' = R_i$  for some  $i$ . Thus  $V \in \text{Rees } I^{R_i}$  by [6, Prop. 3.11]. If  $IR'$  is principal, then  $R'$  dominates the blowup of  $I$  along  $V$ , that is,  $R'$  dominates  $V$ . Therefore we have  $R' = V$  and this implies  $V = \text{ord}_R$ .

For Equation 2, we have  $\text{Rees } \mathfrak{m} = \{\text{ord}_R\}$ , and it is true in general that the Rees valuation rings of a product  $IJ$  of nonzero ideals  $I$  and  $J$  includes  $\text{Rees } I \cup \text{Rees } J$  ([16, Prop. 10.4.5]). The reverse inclusion holds in Equation 2 by Equation 1, since  $\mathfrak{m}I$  and  $I$  have the same transform in each  $R_i$ .  $\square$

**Lemma 2.4.** *Let  $(R, \mathfrak{m})$  be a regular local domain with  $\dim R \geq 2$  and let  $I$  be an  $\mathfrak{m}$ -primary ideal of  $R$  with finitely many base points in the first neighborhood. Let  $S$  be a local ring on  $\text{Proj } \overline{R[It]}$  that dominates  $R$ . If either*

- (1)  *$S$  is a UFD, or*
- (2)  *$S$  has torsion divisor class group and either  $\text{ord}_R \in \text{Rees } I$  or  $S \not\subset \text{ord}_R$ ,*

*then  $\mathfrak{m}S$  is principal.*

*Proof.* Assume that  $S$  is a UFD. By way of contradiction, suppose that  $\mathfrak{m}S$  is not principal. We may write  $\mathfrak{m}S = \rho J$ , where  $\rho$  is a nonzero nonunit of  $S$  and  $J$  is a proper ideal of  $S$  with  $\text{ht } J \geq 2$ . Let  $V$  be a Rees valuation ring of  $J$ . The ring  $S$  is on the blowup of  $I$  and the center of  $V$  on  $S$  is of height at least two since it contains  $J$ . Therefore we have  $V \notin \text{Rees } I$ .

Notice that the blowup of  $\mathfrak{m}I$  may be obtained by first blowing up  $I$  and then blowing up the extension of  $\mathfrak{m}$  to the model  $\overline{R[It]}$ . It follows that  $V \in$

Rees  $\mathfrak{m}I$ . By Lemma 2.3, we must have  $V = \text{ord}_R$ . The ideals  $\rho S$  and  $IS$  have the same radical and  $I$  is contained in the maximal ideal of  $V$ . However, this implies that  $\mathfrak{m}V = \rho JV$  is properly contained in the maximal ideal of  $V$ . This contradicts the fact that  $\mathfrak{m}V$  is the maximal ideal of  $V = \text{ord}_R$ . We conclude that  $\mathfrak{m}S$  is principal.

Assume that  $S$  has torsion divisor class group. As in the previous case, suppose that  $\mathfrak{m}S$  is not principal. Since  $\text{ht } \mathfrak{m}S = 1$ , there exists an integer  $n > 0$  such that  $\mathfrak{m}^n S = \rho J$  for some element  $\rho \in S$  and some ideal  $J$  of  $S$ , where either  $J = S$  or  $\text{ht } J \geq 2$ . Since  $S$  is a local domain, invertible ideals in  $S$  are principal. Moreover, if a power of an ideal is invertible, then the ideal is invertible. Thus the condition that  $\mathfrak{m}S$  is not principal implies that  $\mathfrak{m}^n S$  is not principal, so  $J$  is a proper ideal of  $S$ . By the same argument as the previous case, it follows that  $V$  is a Rees valuation of  $\mathfrak{m}^n I$ . By Lemma 2.3, we have  $V = \text{ord}_R$ . This implies that  $S \subset \text{ord}_R$  and that  $\text{ord}_R \notin \text{Rees } I$ , which completes the proof.  $\square$

Let  $S$  be a local unique factorization domain on the blowup of a finitely supported ideal of a regular local domain  $R$ . In Theorem 2.5, we generalize to the case where  $\dim R > 2$  a result of Huneke and Sally [8, Cor. 1.2] and Lipman [9, Prop. 3.1] for the case where  $\dim R = 2$ . If  $\dim R = 2$ , then every 2-dimensional local UFD that birationally dominates  $R$  is on the blowup of a finitely supported ideal of  $R$ .

**Theorem 2.5.** *Let  $(R, \mathfrak{m})$  be a regular local domain with  $\dim R \geq 2$  and let  $I$  be a finitely supported  $\mathfrak{m}$ -primary ideal. Let  $S$  be a local ring on  $\text{Proj } \overline{R[I]}$  that dominates  $R$ . If  $S$  is a UFD, then  $S$  is a regular local ring. Moreover, either  $S = \text{ord}_{R_i}$  for some base point  $R_i$  of  $I$ , or  $S$  is an infinitely near point to  $R$ .<sup>2</sup>*

*Proof.* By Lemma 2.4(1),  $\mathfrak{m}S$  is principal, so there exists a unique regular local ring  $R'$  in the first neighborhood of  $R$  such that  $S$  dominates  $R'$ . If  $IR'$  is principal, then  $R'$  dominates a unique local ring on  $\text{Proj } \overline{R[It]}$ . Since  $S$  dominates  $R'$ , we have  $R' = S$ . If  $\dim R' = 1$ , then  $R' = \text{ord}_R = S$  while if  $\dim R' \geq 2$ , then  $R' = S$  is an infinitely near point to  $R$  in the first neighborhood.

If  $IR'$  is not principal, then the transform  $I' := I^{R'}$  is a proper ideal, so  $R'$  is a base point of  $I$  in the first neighborhood of  $R$ . It follows that  $I'$  is primary for the maximal ideal  $\mathfrak{m}'$  of  $R'$  [10, Prop. 1.21] and is finitely supported with  $\mathcal{BP}(I')$  a proper subset of  $\mathcal{BP}(I)$ . Also  $S$  dominates  $R'$  and  $S \in \text{Proj } \overline{R'[I't]}$ . The assertions

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<sup>2</sup>By definition, an infinitely near point has dimension at least two.

in Theorem 2.5 therefore follow by a straightforward induction argument on the number of base points of  $I$ .  $\square$

**Discussion 2.6.** Let  $I$  be a finitely supported  $\mathfrak{m}$ -primary ideal of a regular local domain  $(R, \mathfrak{m})$  with  $\dim R = d \geq 2$ . Let  $\Gamma := \mathcal{BP}(I) = \{R = R_0, R_1, \dots, R_n\}$  denote the base points of  $I$ . We observe the following:

- (1) The set  $\text{Rees } I$  of Rees valuation rings of  $I$  is a nonempty subset of the set  $\{\text{ord}_{R_i}\}_{i=0}^n$ , cf. [5, Prop. 4.3].
- (2) The DVRs  $V \in \text{Proj } \overline{R[It]}$  that dominate  $R$  are precisely the DVRs  $V \in \text{Rees } I$  [16, Theorem 10.2.2(3)].
- (3) The DVRs in  $X_\Gamma$  that dominate  $R$  are precisely the DVRs in the set  $\{\text{ord}_{R_i}\}_{i=0}^n$  (see Discussion 1.6).
- (4) Items 2 and 3 imply that the morphism  $f : X_\Gamma \rightarrow \text{Proj } \overline{R[It]}$  of Diagram 1.1 is an isomorphism on dimension one local rings if and only if  $\text{ord}_{R_i} \in \text{Rees } I$  for each base point  $R_i$  of  $I$ .

Therefore if  $f : X_\Gamma \rightarrow \text{Proj } \overline{R[It]}$  is an isomorphism, then the order valuation ring of each base point of  $I$  is a Rees valuation ring of  $I$ .

**Remark 2.7.** Let  $I$  be a finitely supported  $\mathfrak{m}$ -primary ideal of a regular local domain  $(R, \mathfrak{m})$  and let  $S \in \text{Proj } \overline{R[It]}$  be a local domain of dimension at least 2 that dominates  $R$ . Since  $S \in \text{Proj } \overline{R[It]}$ , the ring  $S$  is not a base point of  $I$ . Let  $f : X_\Gamma \rightarrow \text{Proj } \overline{R[It]}$  be the morphism in Diagram 1.1. The local domains  $T \in X_\Gamma$  such that  $f(T) = S$  are regular local domains. Each such  $T$  is either infinitely near to  $R$  or  $T = \text{ord}_{R_i}$  for some base point  $R_i$  of  $I$ . For each  $T$  there exist injective local homomorphisms  $R \hookrightarrow S \hookrightarrow T$ . Since  $T$  is birational over  $S$ , we have  $\dim S \geq \dim T$  [11, Theorem 15.5]. If  $S \neq T$  and  $\dim S = \dim T$ , then Zariski's Main Theorem [12, (37.4)] implies that  $\mathfrak{m}_S T$  is not primary for the maximal ideal of  $T$ , where  $\mathfrak{m}_S$  is the maximal ideal of  $S$ . Hence there exists a nonmaximal prime ideal  $P$  of  $T$  such that  $\mathfrak{m}_S \subset P$ , and  $T_P \in X_\Gamma$  is a regular local ring in the fiber of  $f$  over  $S$  with  $\dim T_P < \dim S$ . If  $\dim S = 2$ , then  $T_P$  is a DVR in  $X_\Gamma$  that dominates  $R$ . This implies that  $T_P = \text{ord}_{R_i}$ , where  $R_i$  is one of the finitely many base points of  $I$ . Thus we have

- (1) Let  $\Gamma := \mathcal{BP}(I) = \{R = R_0, R_1, \dots, R_n\}$  denote the base points of  $I$ . If, as in Discussion 2.6.4, we have  $\text{ord}_{R_i} \in \text{Proj } \overline{R[It]}$  for each  $R_i \in \mathcal{BP}(I)$ , then



the morphism  $f : X_\Gamma \rightarrow \text{Proj } \overline{R[It]}$  is biregular at each  $S \in \text{Proj } \overline{R[It]}$  with  $\dim S \leq 2$  and the singular locus of  $\text{Proj } \overline{R[It]}$  has codimension at least 3.

- (2) If  $\dim R = 3$  and each  $\text{ord}_{R_i}$  is a Rees valuation ring of  $I$ , then  $\text{Proj } \overline{R[It]}$  has only finitely many singular points.

Without assuming that each  $\text{ord}_{R_i}$  is a Rees valuation of  $I$ , let  $S \in \text{Proj } \overline{R[It]}$  be a local domain of dimension at least 2 that dominates  $R$ . Then  $S \in X_\Gamma$  if and only if  $S$  is infinitely near to  $R$ . This follows because each local domain  $T \in X_\Gamma$  that dominates  $R$  is infinitely near to  $R$  and if  $T$  dominates  $S$ , then the unique finite sequence of local quadratic transforms of  $R$  to  $T$  goes through  $S$  if and only if  $S$  is infinitely near to  $R$ . Thus if  $S$  is infinitely near to  $R$  and  $T \in X_\Gamma$  dominates  $S$ , then either  $S = T$  or  $T \neq S$  is infinitely near to  $S$ . But if  $T$  is infinitely near to  $S$ , then  $S$  must be one of the base points  $R_i$ , which it is not since  $S \in \text{Proj } \overline{R[It]}$ .

We conclude that  $f : X_\Gamma \rightarrow \text{Proj } \overline{R[It]}$  is biregular at  $S$  for every local ring  $S \in \text{Proj } \overline{R[It]}$  that is infinitely near to  $R$ , and the following are equivalent:

- (i) The morphism  $f : X_\Gamma \rightarrow \text{Proj } \overline{R[It]}$  is an isomorphism.
- (ii) Each local ring  $S$  on  $\text{Proj } \overline{R[It]}$  with  $\dim S \geq 2$  that dominates  $R$  is an infinitely near point to  $R$ .

These equivalent conditions imply that  $\text{Proj } \overline{R[It]}$  is regular.

**Corollary 2.8.** *Let  $(R, \mathfrak{m})$  be a regular local domain with  $\dim R \geq 2$  and let  $I$  be a finitely supported  $\mathfrak{m}$ -primary ideal. If each local ring  $S \in \text{Proj } \overline{R[It]}$  is a unique factorization domain, then  $\text{Proj } \overline{R[It]}$  is regular and  $I$  has a saturated factorization.*

*Proof.* By Theorem 2.5 every local ring  $S \in \text{Proj } \overline{R[It]}$  that dominates  $R$  is either  $\text{ord}_{R_i}$  for a base point  $R_i$  of  $I$  or is an infinitely near point to  $R$ . By Remarks 2.7, the morphism  $f : X_\Gamma \rightarrow \text{Proj } \overline{R[It]}$  is an isomorphism. Thus  $I$  has a saturated factorization.  $\square$

Theorem 2.5 and Discussion 2.6 also imply the following.

**Corollary 2.9.** *Let  $(R, \mathfrak{m})$  be a regular local domain with  $\dim R \geq 2$  and let  $I$  be a finitely supported  $\mathfrak{m}$ -primary ideal.*

- (1) *If  $R_i \in \mathcal{BP}(I)$  is such that  $\text{ord}_{R_i} \notin \text{Rees } I$ , then for each regular local ring  $T$  on  $X_\Gamma$  such that  $T \subseteq \text{ord}_{R_i}$ , the local ring on  $\text{Proj } \overline{R[It]}$  dominated by  $T$  is not a UFD. Thus the local rings on  $\text{Proj } \overline{R[It]}$  dominated by  $\text{ord}_{R_i}$  or*

dominated by an infinitely near point in the first neighborhood of  $R_i$  on  $X_\Gamma$  are not UFDs.

- (2) If  $R_i \in \mathcal{BP}(I)$  is such that  $\text{ord}_{R_i} \notin \text{Rees } I$ , and if  $S \in \text{Proj } \overline{R[It]}$  is such that  $S \subset \text{ord}_{R_i}$ , then  $S$  is singular.
- (3) If  $\text{Proj } \overline{R[It]}$  is regular, then  $\text{ord}_{R_i} \in \text{Rees } I$  for each  $R_i \in \mathcal{BP}(I)$ .

*Proof.* The statement about  $\text{ord}_{R_i}$  in Item 1 follows directly from Theorem 2.5. For a local ring  $T$  on  $X_\Gamma$  such that  $T \subset \text{ord}_{R_i}$ , let  $S$  denote the local ring on  $\text{Proj } \overline{R[It]}$  dominated by  $T$ . The localization of  $S$  at the center of  $\text{ord}_{R_i}$  is equal to the local ring on  $\text{Proj } \overline{R[It]}$  dominated by  $\text{ord}_{R_i}$ . Thus a localization of  $S$  is not a UFD, so  $S$  is not a UFD.

Items 2 and 3 follow directly from Item 1. □

### 3. TORSION DIVISOR CLASS GROUP ON NORMALIZED BLOWUPS

Let  $(R, \mathfrak{m})$  be a regular local domain with  $\dim R \geq 2$  and let  $I$  be a finitely supported  $\mathfrak{m}$ -primary ideal. Let  $S$  be a local ring on  $\text{Proj } \overline{R[It]}$ . In view of Theorem 2.5 it is natural to ask if  $S$  having torsion divisor class group implies  $S$  is regular. This fails in general as we demonstrate in Example 5.3. With additional assumptions about the Rees valuation rings of  $I$ , we show in Theorem 3.2 that if  $S \in \text{Proj } \overline{R[It]}$  has torsion divisor class group, then  $S$  is regular. We use terminology as in Definition 3.1

**Definition 3.1.** Let  $A$  be an integral domain and let  $B$  be an overring of  $A$  with the same field of fractions. The overring  $B$  is a *sublocalization* of  $A$  if  $B$  is an intersection of localizations of  $A$ . Thus  $B$  is a sublocalization of  $A$  if and only if there exists a family  $\{S_\lambda\}_{\lambda \in \Lambda}$  of multiplicatively closed subsets of nonzero elements of  $A$  such that  $B = \bigcap_{\lambda \in \Lambda} A_{S_\lambda}$ . It is well known that a sublocalization  $B$  of  $A$  is an intersection of localizations of  $A$  at prime ideals. Indeed, for a family  $\{S_\lambda\}_{\lambda \in \Lambda}$  of multiplicatively closed subsets of nonzero elements of  $A$ , we have

$$\bigcap_{\lambda \in \Lambda} A_{S_\lambda} = \bigcap \{A_P \mid P \in \text{Spec } A \text{ and } P \cap S_\lambda = \emptyset \text{ for some } \lambda \in \Lambda\}.$$

**Theorem 3.2.** Let  $(R, \mathfrak{m})$  be a regular local domain with  $\dim R \geq 2$  and let  $I$  be a finitely supported  $\mathfrak{m}$ -primary ideal. Let  $\Gamma := \mathcal{BP}(I)$  denote the set of base points of  $I$ . Let  $Y$  be a normal complete model over  $\text{Spec } R$  that makes the following diagram

commute, where  $f : X_\Gamma \rightarrow \text{Proj } \overline{R[It]}$  is as in Diagram 1.1:

$$\begin{array}{ccc} X_\Gamma & \xrightarrow{f} & \text{Proj } \overline{R[It]} \\ & \searrow g \quad \nearrow h & \\ & Y & \end{array}$$

If  $\text{ord}_{R_i} \in \text{Rees } I$  for each  $R_i \in \Gamma$ , then we have

- (1) For each local domain  $S \in Y$  and each  $T \in g^{-1}(S)$ , the ring  $T$  is a sublocalization over  $S$ .
- (2) The morphism  $g : X_\Gamma \rightarrow Y$  is biregular at each  $S \in Y$  that has torsion divisor class group.
- (3) If  $S \in Y$  is not regular, then the divisor class group of  $S$  is not a torsion group.

*Proof.* Let  $T \in g^{-1}(S)$  and let  $A = h(S)$  be the local ring on  $\text{Proj } \overline{R[It]}$  dominated by  $S$ . We have injective birational local homomorphisms  $A \hookrightarrow S \hookrightarrow T$  of normal Noetherian local domains. We prove that  $T$  is a sublocalization of  $S$ . Since  $S$  and  $T$  are normal Noetherian domains, it suffices to show that  $T_{\mathfrak{p}} = S_{\mathfrak{p} \cap S}$  for each height one prime  $\mathfrak{p}$  of  $T$ . By construction of  $X_\Gamma$ , either  $T_{\mathfrak{p}} = R_{\mathfrak{p} \cap R}$  or  $T_{\mathfrak{p}} = V_i$  for some  $V_i = \text{ord}_{R_i}$ , where  $R_i \in \Gamma$ . In the case where  $T_{\mathfrak{p}} = R_{\mathfrak{p} \cap R}$ , it follows that  $T_{\mathfrak{p}} = S_{\mathfrak{p} \cap S}$ . In the case where  $T_{\mathfrak{p}} = V_i$ , let  $\mathfrak{m}_{V_i}$  denote the maximal ideal of  $V_i$ . Since  $V_i$  is a Rees valuation ring of  $I$ , it follows that  $A_{\mathfrak{m}_{V_i} \cap A} = V_i$ . Thus  $V_i = A_{\mathfrak{m}_{V_i} \cap A} \subseteq S_{\mathfrak{m}_{V_i} \cap S} \subseteq T_{\mathfrak{p}} = V_i$ . Noting that  $\mathfrak{m}_{V_i} \cap S = \mathfrak{p} \cap S$ , it follows that  $\mathfrak{p} \cap S$  is a height 1 prime of  $S$ . Therefore  $T$  is a sublocalization of  $S$ . This proves item 1.

If  $S$  has torsion divisor class group, then every sublocalization of  $S$  is a localization of  $S$ , cf. [7, Cor. 2.9]. Since  $S$  and  $T$  are local and  $S \hookrightarrow T$  is a local homomorphism, if  $T$  is a localization of  $S$ , then  $S = T$ . This proves item 2.

Item 3 is the contrapositive of Item 2. □

**Corollary 3.3.** *Assume the notation of Theorem 3.2. If  $S \in \text{Proj } \overline{R[It]}$  is contained in  $\text{ord}_{R_i}$  for at most one  $i$ , then  $f$  is biregular at  $S$ .*

*Proof.* If  $S$  is not contained in any  $\text{ord}_{R_i}$ , then  $S$  is a localization of  $R$  and there is nothing to show, so assume  $S$  is contained in  $\text{ord}_{R_i}$  for some fixed  $i$ . It follows that  $IS$  is principal, say  $IS = aS$ , and  $aS$  has only one minimal prime  $\mathfrak{p}$ , where  $\mathfrak{p}$  is the center of  $\text{ord}_{R_i}$ . Therefore  $aS = \mathfrak{p}^{(n)}$  for some positive integer  $n$ . Since  $S[\frac{1}{a}]$  is a localization of  $R$ , it is a UFD, so the divisor class group of  $S$  is generated by the

classes of minimal primes of  $aS$ . Therefore the divisor class group of  $S$  is torsion, so the claim follows from Theorem 3.2.2.  $\square$

**Remark 3.4.** Lemma 2.4 can be used to give an alternative proof of item 2 of Theorem 3.2 by an argument along the same lines as the proof given for Theorem 2.5.

**Discussion 3.5.** Let  $(R, \mathfrak{m})$  be a regular local domain with  $\dim R \geq 2$  and let  $I$  be a finitely supported  $\mathfrak{m}$ -primary ideal. Let  $\Gamma := \mathcal{BP}(I)$  denote the set of base points of  $I$  and let  $V_i := \text{ord}_{R_i}$  for each  $R_i \in \Gamma$ . In view of Theorem 3.2, we are motivated to ask for conditions on  $I$  that imply  $\text{ord}_{R_i} \in \text{Rees } I$  for each  $R_i \in \Gamma$ . Lipman's unique factorization theorem for finitely supported complete ideals implies that  $I$  has a factorization as a product of special  $*$ -simple complete ideals with possibly some negative exponents. For each terminal base point  $R_n$  the special  $*$ -simple ideal  $P_{RR_n}$  must occur with a positive exponent. Since  $V_n \in \text{Rees } P_{RR_n}$ , it follows that  $V_n = \text{ord}_{R_n} \in \text{Rees } I$  for each terminal base point  $R_n$  of  $I$ . For each  $R_j \in \Gamma$ , the DVR  $V_j = \text{ord}_{R_j}$  dominates a unique local domain  $S_j \in \text{Proj } \overline{R[It]}$ . Theorem 2.5 implies that  $S_j = V_j$  if  $S_j$  is a UFD. Hence  $V_j \in \text{Rees } I$  in this case.

In Example 6.6, we present an example where  $\text{ord}_{R_i} \in \text{Rees } I$  for each  $R_i \in \Gamma$  and  $\text{Proj } \overline{R[It]}$  has precisely one singular point.

Fix a local domain  $S \in \text{Proj } \overline{R[It]}$  that dominates  $R$ . We observe the following:

- (1) Since  $I$  is  $\mathfrak{m}$ -primary, the minimal primes  $P$  of  $IS$  are the same as the minimal primes of  $\mathfrak{m}S$ . Since  $IS$  is principal, each minimal prime  $P$  of  $IS$  has  $\text{ht } P = 1$ . Since  $S$  is normal,  $S_P = V$  is a DVR and  $V \in \text{Rees } I$ . Thus the association of a minimal prime  $P$  of  $IS$  or  $\mathfrak{m}S$  with the localization  $S_P = V$  yields a one-to-one correspondence between the minimal primes  $P$  of  $I$  and the DVRs  $V \in \text{Rees } I$  such that  $V$  contains  $S$ .
- (2) Let  $f : X_\Gamma \rightarrow \text{Proj } \overline{R[It]}$  be as in Diagram 1.1. The morphism  $f$  is either biregular at  $S$  or the fiber  $f^{-1}(S)$  is infinite and contains both local domains  $T$  with  $\dim T = \dim S$  and local domains  $T$  with  $\dim T < \dim S$ . To see that there exists  $T \in f^{-1}(S)$  with  $\dim T = \dim S$ , let  $(0) = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{m}_S$  be a strictly ascending chain of prime ideals of  $S$  of length equal to  $\dim S$ . By [12, (11.9)], there exists a valuation domain  $W$  that has prime ideals lying over each prime ideal in this chain. Let  $T$  be the local ring on  $X_\Gamma$  dominated by  $W$ . Then  $T \in f^{-1}(S)$  and we have  $\dim T \geq \dim S$  since  $T$  contains a chain of prime ideals of length  $\dim S$  that contract in  $S$  to

distinct prime ideals. Since  $S$  is Noetherian, we also have  $\dim T \leq \dim S$ , so  $\dim T = \dim S$ .

Assume that  $S \neq T$  and let  $I^*$  be an ideal in  $R$  such that  $I^*$  has a saturated factorization and  $\mathcal{BP}(I^*) = \mathcal{BP}(I)$ . Let  $a \in I^*$  be such that  $aT = I^*T$  and let  $A := S[I^*/a]$ . We have  $S \hookrightarrow A \hookrightarrow T$  and  $T$  is a localization of  $A$  at a maximal ideal  $P$ , where  $P \cap S = \mathfrak{m}_S$  is the maximal ideal of  $S$ . Since  $R[I^*/a] \subset S[I^*/a]$ , we have  $A_Q \in X_\Gamma$  for each  $Q \in \operatorname{Spec} A$ . As in Remark 2.7, the ideal  $\mathfrak{m}_S T$  is contained in a nonmaximal prime ideal of  $T$ . Hence there exists a prime ideal  $Q$  of  $A$  such that  $Q \cap S = \mathfrak{m}_S$  and  $Q \subsetneq P$ . Thus the ring  $A/\mathfrak{m}_S A$  has positive Krull dimension, and is a finitely generated algebra over the residue field of  $S$ . Therefore  $\operatorname{Spec}(A/\mathfrak{m}_S A)$  is infinite and hence the fiber  $f^{-1}(S)$  is infinite.

- (3) Since  $S$  is a normal local domain,  $S$  is the intersection of the valuation domains  $W$  that birationally dominate  $S$ , cf [10, Prop. 1.1]. Each of these valuation domains  $W$  dominates a regular local domain  $T \in f^{-1}(S)$ . It follows that  $S = \bigcap \{T \mid T \in f^{-1}(S)\}$ .

#### 4. IDEALS THAT HAVE A SATURATED FACTORIZATION

**Discussion 4.1.** Let  $(R, \mathfrak{m})$  be a regular local ring and let  $I$  be a finitely supported complete  $\mathfrak{m}$ -primary ideal. Let  $\mathcal{BP}(I)$  be the base points of  $I$  and enumerate the base points as  $R = R_0, R_1, \dots, R_n$ . For  $i \in \{0, \dots, n\}$ , let  $P_i$  denote the special  $*$ -simple ideal of  $R$  associated to the pair  $R \prec R_i$ . We consider the following properties the ideal  $I$  may have. Each of the enumerated properties implies that  $I$  has a saturated factorization, that is  $\operatorname{Proj} \overline{R[It]}$  is regular and is equal to  $X_\Gamma$ .

- (1) The product  $P_0 * P_1 * \dots * P_n$  divides  $I$  in the sense that there exists an ideal  $J$  of  $R$  such that  $P_0 * P_1 * \dots * P_n * J = I$ .
- (2) For each  $R_i \in \mathcal{BP}(I)$ , the special star-simple ideal  $P_i$  divides  $I$ .
- (3) The product  $P_0 * P_1 * \dots * P_n$  divides  $I^k$ , for some positive integer  $k$ .
- (4) For each  $S \in \mathcal{BP}(I)$ , the complete transform  $\overline{I^S}$  of  $I$  in  $S$  is divisible by the maximal ideal  $\mathfrak{m}_S$  of  $S$ , that is  $\overline{I^S} = \mathfrak{m}_S * J$  for some ideal  $J \subset S$ .
- (5) There exists a positive integer  $k$  such that for each  $S \in \mathcal{BP}(I)$ , the complete transform  $\overline{(I^k)^S}$  of  $I^k$  in  $S$  is divisible by the maximal ideal  $\mathfrak{m}_S$  of  $S$ , that is  $\overline{(I^k)^S} = \mathfrak{m}_S * J$  for some ideal  $J \subset S$ .

It is straightforward to see that  $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (5)$  and  $(4) \Rightarrow (5)$ . Since the ideals  $I$  and  $I^k$  have the same normalized blowup, and since complete transforms and  $*$ -products commute, Condition 5 implies that  $\text{Proj } \overline{R[It]} = X_\Gamma$ .

Example 4.2 demonstrates the existence of a finitely supported complete ideal of a regular local domain that satisfies Condition 2 but fails to satisfy Condition 1 of Discussion 4.1.

**Example 4.2.** Let  $R$  be a 3-dimensional regular local ring with maximal ideal  $\mathfrak{m} = (x, y, z)R$ . Consider the following infinitely near points  $R_i$  of  $R$ :

$$R := \begin{array}{cc} R_0 & \begin{array}{cc} \prec^x & R_1 \\ \prec^z & R_3 \end{array} \\ & \begin{array}{cc} \prec^{xy} & R_2 \\ \prec^{zy} & R_4 \end{array} \end{array}.$$

Thus  $R_1$  and  $R_3$  are in the first neighborhood of  $R$  and  $R_2$  and  $R_4$  are in the second neighborhood of  $R$ .

The special  $*$ -simple ideals  $P_i$  associated to the pairs  $R \prec R_i$  are

$$\begin{aligned} P_0 &= \mathfrak{m}, \\ P_1 &= (x^2, y, z)R, \\ P_2 &= (x^3, x^2y, xz, y^2, yz, z^2)R, \\ P_3 &= (z^2, x, y)R, \\ P_4 &= (z^3, z^2y, zx, y^2, yx, x^2)R. \end{aligned}$$

The product  $P_2 * P_4 = P_2 P_4$  is divisible by  $\mathfrak{m}^2$  and has a factorization  $J * \mathfrak{m}^2 = J \mathfrak{m}^2$ , where  $J := (xz, y^2, z^3, yz^2, x^2y, x^3)R$ . By an argument similar to [6, Example 4.18], the ideal  $J$  is a  $*$ -simple ideal that is not a special  $*$ -simple ideal. The ideal  $J$  has two Rees valuations,  $\text{Rees } J = \{\text{ord}_{R_2}, \text{ord}_{R_4}\}$ , the order valuations of  $R_2$  and  $R_4$ . Consider the ideal

$$\begin{aligned} I &:= J * \mathfrak{m} * P_1 * P_3 = J \mathfrak{m} P_1 P_3 \\ &= (xyz^3, x^2z^3, y^3z^2, xy^2z^2, x^2yz^2, x^3z^2, y^4z, xy^3z, x^2y^2z, x^3yz, y^5, xy^4, \\ &\quad x^2y^3, yz^5, xz^5, y^2z^4, x^5z, x^4y^2, x^5y, z^7, x^7)R. \end{aligned}$$

Each of the ideals  $P_2$  and  $P_4$  divides  $I$ , so  $I$  satisfies Condition 2 of Discussion 4.1. Since  $\text{ord}_R(P_1 * P_2 * P_3 * P_4) = 6 > \text{ord}_R I = 5$ , the  $*$ -product  $P_1 * P_2 * P_3 * P_4$  does not divide  $I$ . Hence, a fortiori,  $\mathfrak{m} * P_1 * P_2 * P_3 * P_4$  does not divide  $I$ , so the ideal  $I$  does not satisfy Condition 1 of Discussion 4.1.

In Example 7.1, we examine singularities of the \*-simple monomial ideal  $J$  of Example 4.2.

## 5. BLOWUPS OF IDEALS WITH ONLY TWO BASE POINTS

We consider in this section the case where a finitely supported ideal has two base points and no residue field extension.

**Setting 5.1.** Let  $(R, \mathfrak{m})$  be a regular local domain with  $d = \dim R \geq 2$  and let  $R_1$  be an infinitely near point to  $R$  in the first neighborhood. Assume there is no residue field extension from  $R$  to  $R_1$ . By appropriately choosing a regular system of parameters for  $R$ , we may assume that

$$\mathfrak{m} = (x, y_1, \dots, y_{d-1})R \quad \text{and} \quad R_1 = R[\frac{y_1}{x}, \dots, \frac{y_{d-1}}{x}]_{(x, \frac{y_1}{x}, \dots, \frac{y_{d-1}}{x})} R[\frac{y_1}{x}, \dots, \frac{y_{d-1}}{x}].$$

The special \*-simple ideal associated to  $R$  as an infinitely near point to itself is the maximal ideal  $\mathfrak{m}$  of  $R$ . The special \*-simple ideal associated to  $R \prec R_1$  is  $P_1 = (x^2, y_1, \dots, y_{d-1})R$ .

**Discussion 5.2.** With notation as in Setting 5.1, let  $\Gamma := \{R_0, R_1\}$ . For finitely supported ideals  $I$  with  $\mathcal{BP}(I) = \Gamma$ , we observe that there are precisely two possibilities for the model  $\text{Proj } \overline{R[It]}$ . By [6, Theorem 5.4] and the unique factorization theorem of Lipman [10, Theorem 2.5], the complete ideals  $I$  such that  $\mathcal{BP}(I) = \{R_0, R_1\}$  have the form  $I = \mathfrak{m}^i * P_1^j$ , where  $i$  is a nonnegative integer and  $j$  is a positive integer.

- (1) Assume in the factorization  $I = \mathfrak{m}^i * P_1^j$  that the integer  $i$  is positive. Then  $I$  has a saturated factorization and  $\text{Proj } R[It] = X_\Gamma$ . We may take  $i = j = 1$ . The model  $X_\Gamma$  has infinitely near points to  $R$  in the first and second neighborhoods.
- (2) The ideals  $P_1^j$  for  $j$  a positive integer all have the same blowup. Thus one may assume  $j = 1$ . We examine the model  $\text{Proj } R[P_1t]$  in Example 5.3.

**Example 5.3.** Assume notation as in Setting 5.1. The order valuation domain  $\text{ord}_R$  is not in Rees  $P_1$ . By Corollary 2.9, the local ring  $S \in \text{Proj } R[P_1t]$  dominated by  $\text{ord}_R$  is not a UFD. We show the following.

**Fact.** The morphism  $f : X_\Gamma \rightarrow \text{Proj } R[P_1t]$  as in Diagram 1.1 is biregular at the local rings  $T$  on  $\text{Proj } R[P_1t]$  such that  $S$  is not a localization of  $T$ , i.e. such that  $T \not\subset \text{ord}_R$ . Using the language of schemes, let  $\mathfrak{p} \in \text{Proj } R[P_1t]$  be the point corresponding to the local domain  $S$ , that is,  $\mathcal{O}_{\text{Proj } R[P_1t], \mathfrak{p}} = S$ . Then  $f$  induces an

isomorphism on the open sets

$$X_\Gamma \setminus f^{-1}(\overline{\{\mathfrak{p}\}}) \rightarrow \text{Proj } R[P_1t] \setminus \overline{\{\mathfrak{p}\}},$$

where  $\overline{\{\mathfrak{p}\}}$  denotes the Zariski closure of the point  $\mathfrak{p}$ .

To establish the fact stated above, let  $T \in \text{Proj } R[P_1t]$  be a local ring birationally dominating  $R$  such that  $T \not\subset \text{ord}_R$ . We show that  $f$  is biregular at  $T$ . Let  $\mathfrak{q}$  denote the center of  $\text{ord}_{R_1}$  on  $T$  and let  $P_1T = aT = \mathfrak{q}^{(2)}$ . For each height 1 prime ideal  $\mathfrak{p}$  of  $T$ , we have  $T_{\mathfrak{p}}$  is either  $R_{\mathfrak{p} \cap R}$  or  $\text{ord}_{R_1}$ . Since  $T[\frac{1}{a}]$  is a Noetherian normal domain and  $\mathfrak{q}$  is the unique minimal prime ideal of  $aT$ , the ring  $T[\frac{1}{a}]$  is a sublocalization of  $R$ . Since  $R$  is a UFD, it follows that  $T[\frac{1}{a}]$  is also a localization of  $R$  ([7, Cor 2.9]). Thus  $T[\frac{1}{a}]$  is a UFD and the divisor class group of  $T$  is generated by the divisor class of  $\mathfrak{q}$ . Since  $\mathfrak{q}^{(2)} = aT$  is principal, it follows that the divisor class group of  $T$  is a torsion group. By Lemma 2.4,  $\mathfrak{m}T$  is principal. Since  $\text{Proj } R[\mathfrak{m}P_1] = X_\Gamma$ , it follows that  $T$  is on  $X_\Gamma$ , so  $T$  is regular and  $f$  is biregular at  $T$ .

We conclude that the fiber with respect to  $f : X_\Gamma \rightarrow \text{Proj } R[P_1t]$  of the singular locus of  $\text{Proj } R[P_1t]$  consists of the rings  $T$  on  $X_\Gamma$  such that  $T \subset \text{ord}_R$ . In particular, every point in the first neighborhood of  $R$  except  $R_1$  is in the fiber of the singular locus of  $\text{Proj } R[P_1t]$ . To see this, let  $\mathfrak{q}$  denote the point corresponding to  $\text{ord}_R$  in  $\text{Proj } R[P_1t]$ . Then  $f^{-1}(\overline{\{\mathfrak{p}\}}) = \overline{\{\mathfrak{q}\}}$ .

In the case where  $\dim R = 2$ , the local domain  $S$  is the unique singular point of  $\text{Proj } R[P_1t]$ . The fiber  $f^{-1}(S)$  consists of the infinitely near points in the first neighborhood of  $R$  other than  $R_1$  and the point  $R[\frac{x^2}{y}, \frac{y}{x}]_{(\frac{x^2}{y}, \frac{y}{x})R[\frac{x^2}{y}, \frac{y}{x}]}$ . Notice that  $R[\frac{x^2}{y}, \frac{y}{x}]_{(\frac{x^2}{y}, \frac{y}{x})R[\frac{x^2}{y}, \frac{y}{x}]}$  is the unique point in the first neighborhood of  $R_1$  that is contained in  $\text{ord}_{R_0}$ . In classical terminology, this point is said to be proximate to  $R_0$ .

In the case where  $\dim R = n \geq 3$ , the local domain  $S$  is no longer the unique singular point of  $\text{Proj } R[P_1t]$ . We have  $\dim S = 2$ , and the singular locus of  $\text{Proj } R[P_1t]$  is of dimension  $n - 2$ .

In Example 5.3, the powers of the maximal ideal of the local domain  $S$  define a valuation ring  $\text{ord}_S$  and  $\text{ord}_S = \text{ord}_{R_0}$ . This motivates us to ask:

**Question 5.4.** Let  $(R, \mathfrak{m})$  be a regular local ring with  $\dim R \geq 3$  and let  $I$  be a finitely supported  $\mathfrak{m}$ -primary ideal of  $R$ . Let  $R'$  be a base point of  $I$  such that  $V = \text{ord}_{R'}$  is not a Rees valuation ring of  $I$ . Let  $(S, \mathfrak{m}_S)$  denote the ring birationally



dominated by  $V$  on  $\text{Proj } \overline{R[It]}$ . If the powers of  $\mathfrak{m}_S$  define a valuation ring  $\text{ord}_S$ , does it follow that  $\text{ord}_S = V$ ?

**Remark 5.5.** Let  $R \hookrightarrow S$  be an injective extension of regular local domains with  $\dim R = \dim S$  and  $S$  birationally dominating  $R$ . If  $\text{ord}_R = \text{ord}_S$ , then it follows from [13, Cor. 2.6] that  $R = S$ .

Proposition 5.6 answers Question 5.4 in the case where  $V = \text{ord}_R$ .

**Proposition 5.6.** *Let  $R$  be a Noetherian local domain such that the powers of its maximal ideal  $\mathfrak{m}_R$  define a valuation. Let  $V = \text{ord}_R$  denote the associated valuation domain. Let  $S$  be a local domain birationally dominating  $R$  such that  $V$  dominates  $S$ . If the powers of the maximal ideal  $\mathfrak{m}_S$  of  $S$  define a valuation, then  $V$  is the order valuation ring  $\text{ord}_S$ .*

*Proof.* Let  $a \in R$ . Since  $S$  dominates  $R$ ,  $\text{ord}_R a \leq \text{ord}_S a$ , and since  $V$  dominates  $S$ , we have  $\text{ord}_S a \leq \text{ord}_R a$ , so  $\text{ord}_R a = \text{ord}_S a$ . Thus  $\text{ord}_R = \text{ord}_S$  on their common field of fractions, so  $V$  is the order valuation ring  $\text{ord}_S$ .  $\square$

## 6. FINITELY SUPPORTED IDEALS HAVING THE SAME REES VALUATIONS

The examples we present in this section have 3 base points with the base points linearly ordered. We describe the blowups of all the complete ideals having precisely these 3 points as base points.

Steven Dale Cutkosky remarks in [2] that a birational morphism between 2-dimensional normal schemes that is an isomorphism in codimension one must be an isomorphism by Zariski's main theorem. In Example 2 on page 37 of [2], Cutkosky presents an example of an infinite set of normal ideals in a 3-dimensional regular local ring that have the same Rees valuations, but have the property that the blowups of the ideals are pairwise distinct. In Example 6.6, we present an example of normal ideals  $J \subset I$  of a 3-dimensional regular local ring  $R$  that have the same Rees valuations, the ideal  $J$  is a multiple of  $I$  and  $\text{Proj } R[Jt] = X_\Gamma$  is regular while  $\text{Proj } R[It]$  has one singular point.

**Setting 6.1.** Let  $(R, \mathfrak{m})$  be a regular local domain with  $d = \dim R \geq 2$ . Let  $\mathfrak{m} = (x, y)R$  if  $d = 2$  and  $\mathfrak{m} = (x, y, z_1, \dots, z_{d-2})$  if  $d \geq 3$  (and if  $d = 3$ , denote  $z = z_1$ ). Consider the following chain of local quadratic transforms

$$R := R_0 \prec^x R_1 \prec^{yx} R_2,$$

where  $R_1$  with maximal ideal  $\mathfrak{m}_1$  is as in Setting 5.1. Thus

$$R_1 = R[\frac{y}{x}]_{(x, \frac{y}{x})} R[\frac{y}{x}] \quad \text{and} \quad \mathfrak{m}_1 = (x, \frac{y}{x}) R_1 \quad \text{if } d = 2,$$

$$R_1 = R[\frac{m}{x}]_{(x, \frac{y}{x}, \frac{z_1}{x}, \dots, \frac{z_{d-2}}{x})} R[\frac{m}{x}] \quad \text{and} \quad \mathfrak{m}_1 = (x, \frac{y}{x}, \frac{z_1}{x}, \dots, \frac{z_{d-2}}{x}) R_1 \quad \text{if } d \geq 3.$$

Then

$$S_2 := R_1[\frac{\mathfrak{m}_1}{y/x}] \quad \text{and} \quad R_2 := (S_2)_{N_2}$$

where  $N_2 := (\frac{x^2}{y}, \frac{y}{x}) S_2$  if  $d = 2$  and  $N_2 := (\frac{x^2}{y}, \frac{y}{x}, \frac{z_1}{y}, \dots, \frac{z_{d-2}}{y}) S_2$  if  $d \geq 3$ .

For  $i \in \{0, 1, 2\}$ , let  $P_i$  denote the special  $*$ -simple ideals associated to the extension  $R_0 \prec R_i$ . We list generators for the ideals  $P_i$  and the values of the variables with respect to the order valuation rings  $\text{ord}_{R_i}$ . If  $d = 2$ , we have

$P_0$	$=$	$(x, y)R = \mathfrak{m}$	$x$	$y$
$P_1$	$=$	$(x^2, y)R$	$\text{ord}_{R_0}$	1
$P_2$	$=$	$(x^3, x^2y, y^2)R,$	$\text{ord}_{R_1}$	1
			$\text{ord}_{R_2}$	2

If  $d \geq 3$ , then

$P_0$	$=$	$(x, y, z_1, \dots, z_{d-2})R = \mathfrak{m}$	$x$	$y$	$z_i$
$P_1$	$=$	$(x^2, y, z_1, \dots, z_{d-2})R$	$\text{ord}_{R_0}$	1	1
$P_2$	$=$	$(x^3, x^2y, x(z_1, \dots, z_{d-2}), (y, z_1, \dots, z_{d-2})^2)R,$	$\text{ord}_{R_1}$	1	2
			$\text{ord}_{R_2}$	2	3

As in [5, Example 6.13] or [6, Cor. 5.9], if  $d \geq 3$ , then Rees  $P_2 = \{\text{ord}_R, \text{ord}_{R_2}\}$ . If  $d = 2$ , then Rees  $P_2 = \{\text{ord}_{R_2}\}$  as in Discussion 2.6.1.

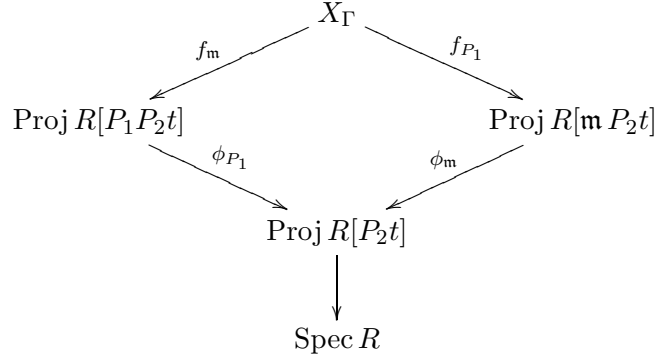
**Discussion 6.2.** With notation as in Setting 6.1, let  $\Gamma := \{R_0, R_1, R_2\}$ . For finitely supported ideals  $I$  with  $\mathcal{BP}(I) = \Gamma$ , we observe that there are precisely 4 possibilities for the model  $\text{Proj } \overline{R[It]}$ . By [6, Theorem 5.4] and the unique factorization theorem of Lipman [10, Theorem 2.5], the complete ideals  $I$  such that  $\mathcal{BP}(I) = \Gamma$  have the form  $I = \mathfrak{m}^i * P_1^j * P_2^k$ , where  $i$  and  $j$  are nonnegative integers and  $k$  is a positive integer. There are the following 4 possible models  $\text{Proj } \overline{R[It]}$ .

- (1) Assume in the factorization  $I = \mathfrak{m}^i * P_1^j * P_2^k$  that  $i$  and  $j$  are both positive. Then  $I$  has a saturated factorization, i.e.,  $\text{Proj } R[It] = X_\Gamma$ . We may take  $i = j = k = 1$ . The ideal  $\mathfrak{m} * P_1 * P_2 = \mathfrak{m} P_1 P_2$  gives the blowup.
- (2) Assume in the factorization  $I = \mathfrak{m}^i * P_1^j * P_2^k$  that  $i > 0$  and  $j = 0$ . The ideals  $\mathfrak{m}^i P_2^k$  for  $i$  and  $k$  positive all have the same blowup. Thus we may assume  $i = k = 1$ . The ideal  $\mathfrak{m} * P_2 = \mathfrak{m} P_2$  gives this blowup.
- (3) Assume in the factorization  $I = \mathfrak{m}^i * P_1^j * P_2^k$  that  $i = j = 0$ . The ideals  $P_2^k$  for  $k$  a positive integer all have the same blowup. Thus one may assume  $k = 1$ . The ideal  $P_2$  gives this blowup.

- (4) Assume in the factorization  $I = \mathfrak{m}^i * P_1^j * P_2^k$  that  $i = 0$  and  $j > 0$ . The ideals  $P_1^j * P_2^k$  with  $j$  and  $k$  both positive all have the same blowup. Thus we may assume  $j = k = 1$ . The ideal  $P_1 * P_2 = P_1 P_2$  gives this blowup.

The four models and the natural morphisms among these models are displayed in Diagram 6.2.

(6.2)



There are significant differences between the case where  $\dim R = 2$  and the case where  $\dim R \geq 3$  that are related to the fact that  $\text{Rees } P_2 = \{\text{ord}_{R_2}\}$  if  $\dim R = 2$  while  $\text{Rees } P_2 = \{\text{ord}_{R_2}, \text{ord}_R\}$  if  $\dim R \geq 3$ . In Example 6.3 we describe the situation where  $\dim R = 2$ .

**Example 6.3.** Assume notation as in Setting 6.1 and that  $\dim R = 2$ . Thus  $P_2 = (x^3, x^2 y, y^2)R$  and  $\text{Proj } R[P_2 t]$  has 2 singular points

$$S_0 := R\left[\frac{x^3}{y^2}, \frac{x^2}{y}\right]_{(x, y, \frac{x^3}{y^2}, \frac{x^2}{y})R[\frac{x^3}{y^2}, \frac{x^2}{y}]} \quad \text{and} \quad S_1 := R\left[\frac{y}{x}, \frac{y^2}{x^3}\right]_{(x, \frac{y}{x}, \frac{y^2}{x^3})R[\frac{y}{x}, \frac{y^2}{x^3}]}$$

The local domain  $S_0 \in \text{Proj } R[P_2 t]$  is dominated by  $\text{ord}_R$  and  $S_1 \in \text{Proj } R[P_2 t]$  is dominated by  $\text{ord}_{R_1}$ . The divisor class group  $\text{Cl}(S_0)$  is a cyclic group of order 3, and the divisor class group  $\text{Cl}(S_1)$  is a cyclic group of order 2. The local domains  $S_0$  and  $S_1$  are both localizations of the affine chart  $R[\frac{P_2}{x^3+y^2}]$  and the divisor class group  $\text{Cl}(R[\frac{P_2}{x^3+y^2}])$  is a cyclic group of order 6.

The local domain  $S_0$  is also on the model  $\text{Proj } R[P_1 P_2 t]$  and is the unique singular point on this model, while the local domain  $S_1$  is on the model  $\text{Proj } R[\mathfrak{m} P_2 t]$  and is the unique singular point on this model.

With notation as in Diagram 6.2, we have:

- (1) The morphism  $\phi_{P_1}$  is an isomorphism off the fiber  $\phi_{P_1}^{-1}(S_1)$ .
- (2) The morphism  $\phi_{\mathfrak{m}}$  is an isomorphism off the fiber  $\phi_{\mathfrak{m}}^{-1}(S_0)$ .

- (3) The morphism  $f_{P_1}$  is an isomorphism off the fiber  $f_{P_1}^{-1}(S_1)$ .
- (4) The morphism  $f_{\mathfrak{m}}$  is an isomorphism off the fiber  $f_{\mathfrak{m}}^{-1}(S_0)$ .

This completes our description of the case where  $\dim R = 2$ .

Assume that  $\dim R = 3$ . In Examples 6.4, 6.5, and 6.6, we consider the models obtained by blowing up the ideals  $\mathfrak{m}P_2$ ,  $P_2$ , and  $P_1P_2$ , respectively.

**Example 6.4.** Assume notation as in Setting 6.1 with  $\dim R = 3$ . Consider the ideal  $I = \mathfrak{m}P_2$  and its blowup

$$\text{Proj } R[It] = \text{Proj } R[x^4t, x^3yt, x^2zt, xy^2t, xyzt, xz^2t, y^3t, y^2zt, yz^2t, z^3t].$$

The transform of  $I$  in  $R_1$  is the ideal

$$I_1 := \left( x, \left( \frac{y}{x} \right)^2, \frac{z}{x} \right) R_1$$

and  $I_1$  is the special  $*$ -simple ideal associated to the pair  $R_1 \prec R_2$ , see [10, Prop. 2.1]. The natural morphism  $\phi_{P_2} : \text{Proj } R[It] \rightarrow \text{Proj } R[\mathfrak{m}t]$  is an isomorphism off the fiber  $\phi_{P_2}^{-1}(R_1)$  of  $R_1$ . Moreover, with  $I_1$  the transform of  $I$  in  $R_1$ , the restriction  $\phi_{P_2} : \text{Proj } R_1[I_1t] \rightarrow \text{Spec } R_1$  is as in Example 5.3. Thus the singular locus of  $\text{Proj } R[It]$  is determined by the center of  $\text{ord}_{R_1}$  on  $\text{Proj } R[It]$ .

**Example 6.5.** Assume notation as in Setting 6.1 with  $\dim R = 3$ . Consider the ideal  $P_2$ , where

$$P_2 = (x^3, x^2y, xz, y^2, yz, z^2)R$$

is the special  $*$ -simple ideal associated to the extension  $R_0 \prec R_2$ . The blowup of  $P_2$  is

$$\text{Proj } R[P_2t] = \text{Proj } R[x^3t, x^2yt, xzt, y^2t, yzt, z^2t].$$

We consider affine charts of  $\text{Proj } R[P_2t]$  and examine their singularities. The ideal  $(y^2, xz, z^2, x^3)R$  is a monomial reduction of  $P_2$ . It suffices to consider the affine charts  $R[\frac{P_2}{\rho}]$ , where  $\rho \in \{y^2, xz, z^2, x^3\}$ . We have the four affine charts:

$$A := R\left[\frac{P_2}{y^2}\right] \quad B := R\left[\frac{P_2}{xz}\right] \quad C := R\left[\frac{P_2}{z^2}\right] \quad D := R\left[\frac{P_2}{x^3}\right].$$

The affine chart  $A = R[\frac{x^3}{y^2}, \frac{x^2}{y}, \frac{xz}{y^2}, \frac{z}{y}]$  is not contained in  $\text{ord}_{R_1}$  since  $\frac{x^3}{y^2}$  and  $\frac{xz}{y^2}$  have negative value for  $\text{ord}_{R_1}$ . By individually inverting each of the generators of  $\mathfrak{m}_A := (x, y, \frac{z}{y}, \frac{xz}{y^2}, \frac{x^2}{y}, \frac{x^3}{y^2})A$  and checking that the ring we obtain is regular, we conclude that  $\mathfrak{m}_A$  is the unique singular point of  $A$ . We compute that  $S := A_{\mathfrak{m}_A}$  is a 3-dimensional Cohen-Macaulay normal local domain of embedding dimension 6

and multiplicity 4 where, for instance,  $(\frac{z}{y}, \frac{x^3}{y^2}, y - \frac{xz}{y^2})S$  is a system of parameters for  $S$ . The ring  $A$  is also an affine chart for  $\text{Proj } R[P_1P_2t]$  and  $S$  is the unique singular point of the model  $\text{Proj } R[P_1P_2t]$ . We examine this in more detail in Example 6.6.

The affine chart  $C = R[\frac{P_2}{z^2}] = R[\frac{x}{z}, \frac{y}{z}]$  is regular.

The affine chart  $D = R[\frac{y}{x}, \frac{z}{x^2}, \frac{y^2}{x^3}]$  is contained in the valuation domain  $\text{ord}_{R_1}$ . The center of  $\text{ord}_{R_1}$  on  $D$  is the height 2 prime ideal  $Q := (x, \frac{y}{x}, \frac{y^2}{x^3})D$ . We compute that  $D_Q$  is a 2-dimensional normal local domain of multiplicity 2. Moreover, the singular locus of  $D$  is the set of prime ideals of  $D$  that contain  $Q$ .

The affine chart  $B = R[\frac{x^2}{z}, \frac{y^2}{xz}, \frac{y}{x}, \frac{z}{x}]$  is also contained in  $\text{ord}_{R_1}$ . The center of  $\text{ord}_{R_1}$  on  $B$  is the height 2 prime ideal  $Q' := (\frac{y^2}{xz}, \frac{y}{x}, \frac{z}{x})B$ . We have  $B_{Q'} = D_Q$ , and compute that the singular locus of  $B$  is the set of prime ideals of  $B$  that contain  $Q'$ .

Since  $\mathfrak{m}B$  and  $\mathfrak{m}D$  are principal, the affine charts  $B$  and  $D$  of  $\text{Proj } R[P_2t]$  are also affine charts of  $\text{Proj } R[\mathfrak{m}P_2t]$  and the morphism  $\phi_{\mathfrak{m}}$  of Diagram 6.2 is an isomorphism on these affine charts.

The local domain on  $\text{Proj } R[P_2t]$  dominated by  $\text{ord}_{R_1}$  is  $B_{Q'} = D_Q$ . The morphism  $\phi_{P_1}$  of Diagram 6.2 is biregular at all the local domains  $S \in \text{Proj } R[P_2t]$  except those  $S$  such that  $B_{Q'} = D_Q$  is a localization of  $S$ , that is, the morphism  $\phi_{P_1}$  is biregular off the center of  $\text{ord}_{R_1}$  on  $\text{Proj } R[P_2t]$ .

**Example 6.6.** Assume notation as in Setting 6.1 with  $\dim R = 3$  and let

$$I = P_1P_2 = (z^3, yz^2, xz^2, y^2z, xyz, y^3, x^3z, x^2y^2, x^3y, x^5)R.$$

Let  $J := \mathfrak{m}P_1P_2 = \mathfrak{m}I$ . By Remark 1.2 the ideal  $J$  has a saturated factorization, i.e.,  $R[Jt] = X_{\Gamma}$ . We have  $\text{Rees } I = \text{Rees } J = \{\text{ord}_{R_0}, \text{ord}_{R_1}, \text{ord}_{R_2}\}$ . We compute that  $\text{Proj } R[It]$  is normal and has precisely one singular point.

We use that  $K := (z^3, xz^2, xyz, y^3, x^3y, x^5)R$  is a monomial reduction of  $I$  and check the affine charts associated to each of the monomial generators of  $K$ . The affine chart  $A := R[\frac{I}{y^3}] = R[\frac{z}{y}, \frac{xz}{y^2}, \frac{x^2}{y}, \frac{x^3}{y^2}]$  is the only affine chart that has a singularity. We compute that the affine chart  $A$  is a 3-dimensional normal domain, and prove below that the maximal ideal  $\mathfrak{m}_A := (\mathfrak{m}, \frac{z}{y}, \frac{xz}{y^2}, \frac{x^2}{y}, \frac{x^3}{y^2})A$  is the unique singular point of  $A$ .

Observe that  $\text{ord}_{R_0}$  and  $\text{ord}_{R_2}$  contain  $A$ , but  $\text{ord}_{R_1}$  does not. The center of  $\text{ord}_{R_0}$  on  $A$  is the height-one prime ideal  $\mathfrak{p}_0 := (\mathfrak{m}, \frac{x^2}{y}, \frac{x^3}{y^2})A$ , and the center of  $\text{ord}_{R_2}$  on  $A$  is the height-one prime ideal  $\mathfrak{p}_2 := (\mathfrak{m}, \frac{z}{y}, \frac{x^2}{y})A$ .

We show that the divisor class group of  $A$  is an infinite cyclic group. Since  $A[\frac{1}{y}] = R[\frac{1}{y}]$ , a theorem of Nagata ([15, Theorem 6.3, p. 17]) implies that the divisor class group  $\text{Cl}(A)$  of  $A$  is generated by the minimal primes of  $yA$ . The minimal primes of  $yA$  and  $y^3A = IA$  are equal. Therefore, we see that  $\text{Min } yA = \{\mathfrak{p}_0, \mathfrak{p}_2\}$ . Since  $\text{ord}_{R_0}(y) = 1$  and  $\text{ord}_{R_2}(y) = 3$  and  $yA$  is an unmixed height 1 ideal, we have  $yA = \mathfrak{p}_0 \cap \mathfrak{p}_2^{(3)}$ . The divisor class group  $\text{Cl}(A)$  is generated by  $[\mathfrak{p}_0], [\mathfrak{p}_2]$ , where  $[\ ]$  represent the class of a height 1 prime ideal in  $\text{Cl}(A)$ . The equality  $yA = \mathfrak{p}_0 \cap \mathfrak{p}_2^{(3)}$  gives a relation  $[\mathfrak{p}_0] = -3[\mathfrak{p}_2]$ , and in fact this is the only relation since  $\text{Cl}(A)$  is not torsion by Theorem 3.2(2). Therefore, we have  $\text{Cl}(A) = \langle [\mathfrak{p}_2] \rangle$ .

To prove that the singular locus of  $A$  is  $\mathfrak{m}_A$ , let  $\mathfrak{q}$  be a prime ideal of  $A$  and consider the localization  $A_{\mathfrak{q}}$ . If  $\mathfrak{q}$  does not contain both  $\mathfrak{p}_0$  and  $\mathfrak{p}_2$ , then by Nagata's Theorem and the relation  $[\mathfrak{p}_0] = -3[\mathfrak{p}_2]$ , the ring  $A_{\mathfrak{q}}$  has torsion divisor class group. Theorem 3.2(2) then implies that  $A_{\mathfrak{q}}$  is regular. Assume that  $\mathfrak{q}$  contains both  $\mathfrak{p}_0$  and  $\mathfrak{p}_2$ . Notice that  $(\mathfrak{p}_0, \mathfrak{p}_2, \frac{xz}{y^2})A = \mathfrak{m}_A$ . Hence if  $\mathfrak{q} \neq \mathfrak{m}_A$ , then  $\frac{xz}{y^2} \notin \mathfrak{q}$ . In  $A[\frac{y^2}{xz}]$ , we have  $\frac{y}{x} = \frac{z}{y} \cdot \frac{y^2}{xz}$  and  $\frac{z}{x} = \frac{y}{x} \cdot \frac{z}{y}$ . This implies that  $\mathfrak{m}_A[\frac{y^2}{xz}] = xA[\frac{y^2}{xz}]$  is principal. In particular, we have  $\mathfrak{m}_A_{\mathfrak{q}}$  is principal. Therefore,  $A_{\mathfrak{q}}$  is on the regular model  $X_{\Gamma}$  and hence is regular.

Assume now that  $\dim R = d \geq 4$  and denote  $\mathfrak{m} = (x, y, z_1, \dots, z_{d-2})$ . The structure of the special  $*$ -simple ideal  $P_2$  is similar to the 3-dimensional case, but with more generators as we increase  $d$ . The minimal number of generators of  $P_2$  is the same as that for  $\mathfrak{m}^2$ . The difference between  $\mathfrak{m}^2$  and  $P_2$  is that  $x^2$  is replaced by  $x^3$  and  $xy$  by  $x^2y$ . Thus if  $\dim R = d$ , then

$$P_2 = (x^3, x^2y, x(z_1, \dots, z_{d-2}), (y, z_1, \dots, z_{d-2})^2)R$$

is minimally generated by  $\binom{d+1}{2}$  elements. We have  $\text{Rees } P_2 = \{\text{ord}_{R_2}, \text{ord}_R\}$ .

As in the case where  $\dim R = 3$ , the affine chart  $A := R[\frac{P_2}{y^2}]$  of  $\text{Proj } R[P_2t]$  contains precisely one prime ideal for which the localization of  $A$  is not regular. We have

$$A = R[\frac{P_2}{y^2}] = R[\frac{x^3}{y^2}, \frac{x^2}{y}, \frac{xz_1}{y^2}, \dots, \frac{xz_{d-2}}{y^2}, \frac{z_1}{y}, \dots, \frac{z_{d-2}}{y}]$$

and  $\mathfrak{m}_A := (x, y, \frac{x^3}{y^2}, \frac{x^2}{y}, \frac{xz_1}{y^2}, \dots, \frac{xz_{d-2}}{y^2}, \frac{z_1}{y}, \dots, \frac{z_{d-2}}{y})A$ . Notice that  $P_1A = yA$ , thus  $A$  is also an affine chart on  $\text{Proj } R[P_1P_2t]$ . We have  $\text{ord}_{R_0}$  and  $\text{ord}_{R_2}$  contain  $A$ , while  $\text{ord}_{R_1}$  does not. The center of  $\text{ord}_{R_0}$  on  $A$  is the height-one prime ideal  $\mathfrak{p}_0 := (x, y, \frac{x^2}{y}, \frac{x^3}{y^2})A$ , and the center of  $\text{ord}_{R_2}$  on  $A$  is the height-one prime ideal  $\mathfrak{p}_2 := (x, y, \frac{x^2}{y}, \frac{z_1}{y}, \dots, \frac{z_{d-2}}{y})A$ .

The proof given above for the case where  $\dim R = 3$  also applies here to show that the divisor class group  $\text{Cl}(A)$  is the infinite cyclic group generated by  $[\mathfrak{p}_2]$ . To prove that  $\mathfrak{m}_A$  is the unique prime ideal of  $A$  at which the localization is not regular, let  $\mathfrak{q} \in \text{Spec } A$  be such that  $A_{\mathfrak{q}}$  is not regular. Since  $A_{\mathfrak{q}}$  must have nontorsion divisor class group by Theorem 3.2,  $\mathfrak{q}$  contains  $\mathfrak{p}_0 + \mathfrak{p}_2$ . The remaining generators of  $\mathfrak{m}_A$  are of the form  $\frac{xu}{y^2}$ , where  $u$  varies among the variables  $z_1, \dots, z_{d-2}$ . If  $\mathfrak{q} \neq \mathfrak{m}_A$ , then by symmetry of the variables  $z_1, \dots, z_{d-2}$ , we may assume  $\frac{xz_1}{y^2} \notin \mathfrak{q}$ . But in  $A[\frac{y^2}{xz_1}]$ , a simple computation as above shows that  $\mathfrak{m}$  extends to a principal ideal. Therefore,  $A_{\mathfrak{q}}$  is on the regular model  $X_{\Gamma}$  and hence is regular. Thus the maximal ideal  $\mathfrak{m}_A$  is the unique singular point of  $A$ .

## 7. SINGULARITIES ON THE BLOWUP OF FINITELY SUPPORTED IDEALS

We are interested in algebraic properties of the singularities of local rings on the normalized blowup of finitely supported ideals. Huneke and Sally in [8] examine the structure of 2-dimensional normal local rings  $S$  that birationally dominate a 2-dimensional regular local ring. Using algebraic techniques, Huneke and Sally recover much information that was known from work of Lipman and Artin about the structure of  $S$  such as that  $S$  has a rational singularity and minimal multiplicity. For example, they show that  $S$  is Gorenstein if and only if  $S$  has multiplicity at most 2 [8, Cor. 1.6].

Example 7.1 is a further discussion of Example 4.2 regarding the blowup of a finitely supported ideal.

**Example 7.1.** Let  $(R, \mathfrak{m})$  be a regular local ring with  $\mathfrak{m} = (x, y, z)$  and let  $J = (xz, y^2, z^3, yz^2, x^2y, x^3)$ . Consider the affine chart of  $\text{Proj } R[Jt]$  obtained by homogeneous localization at the element  $xzt$ . This gives the ring  $A = R[\frac{y^2}{xz}, \frac{z^2}{x}, \frac{yz}{x}, \frac{xy}{z}, \frac{x^2}{z}]$ . We observe below that  $\text{ord}_{R_2}$  and  $\text{ord}_{R_4}$  are centered on height 1 primes of  $A$ , and  $\text{ord}_{R_0}$ ,  $\text{ord}_{R_1}$ , and  $\text{ord}_{R_3}$  are centered on height 2 primes of  $A$ . All of these prime ideals are contained in the maximal ideal  $\mathfrak{m}_A$ , where

$$\mathfrak{m}_A := (x, y, z, \frac{y^2}{xz}, \frac{z^2}{x}, \frac{yz}{x}, \frac{xy}{z}, \frac{x^2}{z})A.$$

Thus  $\mathfrak{m}_A$  is generated by  $\mathfrak{m}$  and the five listed ring generators of  $A$  over  $R$ . The powers of  $\mathfrak{m}_A$  do not define a valuation, since  $y \in \mathfrak{m}_A \setminus \mathfrak{m}_A^2$  and  $y^2 \in \mathfrak{m}_A^3$  by the relation  $xz(\frac{y^2}{xz}) = y^2$ .

Using the chart

	$x$	$y$	$z$
$\text{ord}_{R_0}$	1	1	1
$\text{ord}_{R_1}$	1	2	2
$\text{ord}_{R_2}$	2	3	4
$\text{ord}_{R_3}$	2	2	1
$\text{ord}_{R_4}$	4	3	2,

we compute the centers  $Q_i$  of  $\text{ord}_{R_i}$  on  $A$  for  $i \in \{0, \dots, 4\}$ . They are

$$\begin{aligned}
Q_0 &= (x, y, z, \frac{z^2}{x}, \frac{yz}{x}, \frac{xy}{z}, \frac{x^2}{z})A, \\
Q_1 &= (x, y, z, \frac{y^2}{xz}, \frac{z^2}{x}, \frac{yz}{x}, \frac{xy}{z})A, \\
Q_2 &= (x, y, z, \frac{z^2}{x}, \frac{yz}{x}, \frac{xy}{z})A, \\
Q_3 &= (x, y, z, \frac{y^2}{xz}, \frac{yz}{x}, \frac{xy}{z}, \frac{x^2}{z})A, \text{ and} \\
Q_4 &= (x, y, z, \frac{yz}{x}, \frac{xy}{z}, \frac{x^2}{z})A.
\end{aligned}$$

Since  $\text{Rees } J = \{\text{ord}_{R_2}, \text{ord}_{R_4}\}$ , the prime ideals  $Q_2$  and  $Q_4$  are of height 1 and the prime ideals  $Q_0, Q_1, Q_3$  are of height 2. The ideal  $L := (\frac{z^2}{x}, \frac{x^2}{z}, \frac{y^2}{xz})A$  is a reduction of  $\mathfrak{m}_A$ . Direct computation shows that  $L\mathfrak{m}_A = \mathfrak{m}_A^2$ . Therefore the reduction number of  $\mathfrak{m}_A$  with respect to  $L$  is 1, and the local ring  $A_{\mathfrak{m}_A}$  has minimal multiplicity with Hilbert-Samuel multiplicity  $e(A_{\mathfrak{m}_A}) = 6$ .

We have  $A = R[\frac{1}{xz}] \cap A_{Q_2} \cap A_{Q_4}$ . The divisor class group  $\text{Cl}(A)$  is generated by the classes  $[Q_2]$  and  $[Q_4]$ . Notice that  $\frac{z^2}{x}A = Q_2^{(6)}$  and  $\frac{x^2}{z}A = Q_4^{(6)}$ . Also we have

$$xA = Q_2^{(2)} \cap Q_4^{(4)} \quad \text{and} \quad zA = Q_2^{(4)} \cap Q_4^{(2)}.$$

The localization  $A[(\frac{z^2}{x})^{-1}] = R[\frac{x}{z^2}] = R[\frac{1}{xz}] \cap A_{Q_4}$  is contained in  $\text{ord}_{R_3}$  and hence is not a UFD by Corollary 2.9. Since  $zA[\frac{x}{z^2}] = Q_4^{(2)}A[\frac{x}{z^2}]$ , the divisor class group  $\text{Cl}(A[\frac{x}{z^2}])$  is a cyclic group of order 2, and the divisor class group  $\text{Cl}(A)$  is the direct sum of a cyclic group of order 6 with a cyclic group of order 2.

With  $A$  and  $\mathfrak{m}_A$  as in Example 7.1, we noted above that the powers of the maximal ideal of the local domain  $S := A_{\mathfrak{m}_A}$  do not define a valuation. The ring  $S$  is on the blowup  $\text{Proj } R[It]$  of a finitely supported ideal  $J$  of a regular local ring. It seems natural to ask:

**Question 7.2.** Let  $I$  be a finitely supported ideal of a regular local ring  $R$  and let  $S \in \text{Proj } \overline{R[It]}$ . Under what conditions do the powers of the maximal ideal of  $S$  define a valuation?



Let  $R$  be a regular local ring with  $\dim R \geq 2$ , and let  $\Gamma$  be a finite set of infinitely near points to  $R$  that satisfies the 3 conditions of Remark 1.2 and thus is the set of base points of a finitely supported ideal of  $R$ . We ask:

**Question 7.3.** Among the finitely supported ideals  $I$  of  $R$  with  $\mathcal{BP}(I) = \Gamma$ , how many distinct projective models  $\text{Proj } \overline{R[It]}$  exist? If  $\Gamma$  has 1 terminal point and  $n$  points which are not terminal points, are there precisely  $2^n$  distinct such models?

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